Mécanique statistique classique 11 novembre 2009
Série 5

Problem 1
Let's investigate a toy Ising model where only two spins are present and hence the Hamiltonian of the system is

$$H = -Js_1s_2 - h(s_1 + s_2).$$

a) Using

$$\langle s_1 \rangle = \frac{1}{Z} \sum_{\text{conf}} s_1 \mathrm{e}^{-\beta E(\text{conf})}$$

show that

$$\langle s_1 \rangle = \frac{\mathrm{e}^{\beta J} \sinh(2\beta h)}{\mathrm{e}^{\beta J} \cosh(2\beta h) + \mathrm{e}^{-\beta J}}.$$

b) Find the high and low temperature limit of $\langle s_1 \rangle$. How do they relate to our expectations?
c*) Show how $\langle s_1 s_2 \rangle$ and $\langle s_1 \rangle$ can be derived directly from the partition function Z.

Problem 2
Ising model with an external magnetic field can be described by the Hamiltonian

$$H = -J \sum_{<i,j>} s_i s_j - h \sum_i s_i$$

where h represents the intensity of the magnetic field.
Starting from the definition of the total magnetization $M = \sum_i \langle s_i \rangle$, show that susceptibility

$$\chi = \lim_{h \to 0} \frac{\partial M}{\partial h}$$

can be written as

$$\chi = \frac{1}{k_B T} \left(\sum_{i,j} \langle s_i s_j \rangle_0 - M_0^2 \right)$$

where by index 0 we label values of quantities in the limit $h \to 0$.

Problem 3
Let's study the Ising model in one dimension, without external magnetic field and with periodic boundary conditions. Assuming that there are N spins in the chain.
a) Compute heat capacity C exactly (as in the lecture).
b) Show that in the mean-field approximation, heat capacity is discontinuous but not diverging at T_c. Sketch the course of $C(T)$. What is its limit value at $T \to T_c^+$?
c*) With help of a computer plot these two results in one graph as a function of βJ.